http://people.csail.mit.edu/wangliang/Papers/New%20Way%20of%20Teaching%20Calculus_Lin_Qun.pdf

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Calculus Relies on Guessing (Based on an Image)

New Proof (determined by one-line and QED in two-lines)

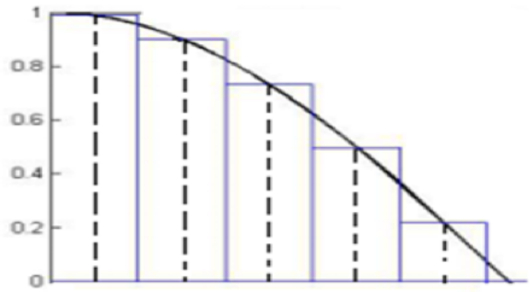
From Qun Lin's blog on ScienceNet (http://blog.sciencenet.cn/blog-1252-1054538.html)

Introduction to Lin's calculus education

April 18, 2018

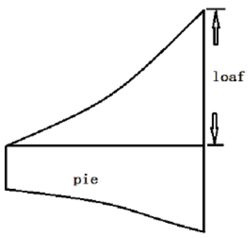
The key problem in calculus is to calculate the area. It is a common mathematic problem, and everyone should know this.

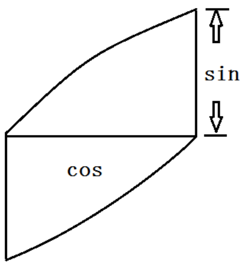
Rectangular area = length $\times$ width. However, how to calculate curved classroom area? It's impossible to cover the area with rectangle floor pieces $\because$ Arc area $\neq$ sum of rectangular areas:

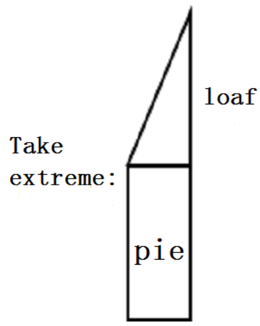


Calculus provides the solution: changing the problem of calculating area as the problem of calculating height.

Area of a pie = height of a Loaf (They are all numbers) (1)



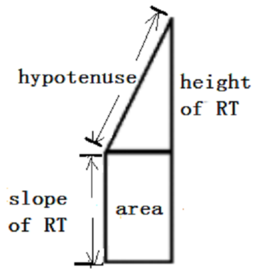




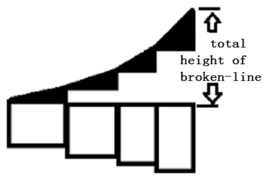
(Area under a curve = height under another curve)

Such a calculus (of one-variable) has only one theorem (pie = loaf) with least concepts(such as slope) and shortest proof(of two-lines).
Guess first, and then prove.

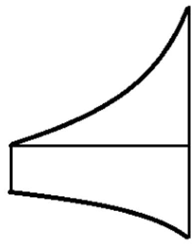
1. Let's guess first: By Imagination



A RT from a Rect



Broken-line RTs from ladder Rects



A curved triangle from a curved trapezoid

1^{st}Fig: In order to guarantee that "loaf=pie", let's recall that

$$\text{Height of RT(loaf)} = \text{Slope of RT} \times \text{bottom} = \text{Area of Rect (with the Height = Slope of RT)}$$

which leads to the construction of Rect(pie): Height of Rect = Slope of RT.
2^{nd}Fig: Height of each RT = Area of each ladder Rect (with the Height = Slope of RT). Sum up

$$\text{Total height of broken-line} = \text{Total area of ladder Rect}$$

3^{rd}Fig (derived to the ultimate,or looking far): Let the broken-line become to the continuous curve, the area of ladder Rect become to the area of "slope curve" (since the height of such a curve = slope of the continuous curve, so we call it to be the slope curve) ,we guess the fundamental theorem

$$\text{Height of a curve (loaf)} = \text{Area enclosed by the slope curve (pie)}$$

It is also possible to look backwards: the slope curve of the 3^{rd}Fig is consisted of the ladder Rect of the 2^{nd}Fig, the curved triangle of the 3^{rd}Fig is consisted of the broken-line of the 2^{nd}Fig, and the 2^{nd}Fig is consisted of the 1^{st}Fig. All returns to the 1^{st}Fig. Once you understand the 1^{st}Fig and you will understand the fundamental theorem.

In short, what the most important is the 1^{st}Fig, where, Height of Rect = Slope of RT.

People need to get used to seeing problems simply!

2. Checking (to achieve the shortest: determined by one-line and QED in two-lines)

Before checking the general theory, let's take an example $y = \sin(x)$.
The height of $\sin(x)$ over the sub-interval $[x, x+\theta]$, $\sin(x+\theta) - \sin(x) = 2\sin(\frac{\theta}{2})\cos(x+\frac{\theta}{2})$, i.e. the Height of RT. Furthermore, the bottom of RT is θ, so the Slope of RT, $\frac{height}{bottom} = \frac{\sin(x+\theta)-\sin(x)}{\theta} = \frac{\sin(\frac{\theta}{2})}{\frac{\theta}{2}}\cos(x+\frac{\theta}{2})$. Don't forget that the Height of Rect= Slope of RT $= \frac{\sin(\frac{\theta}{2})}{\frac{\theta}{2}}\cos(x+\frac{\theta}{2})$. When $\theta \to 0$,the Height of Rect becomes the slope $\cos(x)$: $\frac{\sin(x+\theta)-\sin(x)}{\theta} = \frac{\sin(\frac{\theta}{2})}{\frac{\theta}{2}}\cos(x+\frac{\theta}{2}) \to \cos(x)$. Hence,the Height of Rect forms the slope curve with its curved equation to be $y = \cos(x)$. Without loss of generality, we suppose $\cos(x) > 0$ (see the calculus card next page),then,put right $\cos(x+\frac{\theta}{2})$ to left

$$\frac{\sin(x+\theta)-\sin(x)}{\cos(x+\frac{\theta}{2})\cdot\theta} = \frac{\sin(\frac{\theta}{2})}{\frac{\theta}{2}} \to 1$$

Summing up numerator and denominator

$$\frac{\sin(b)-\sin(a)}{\sum_x \cos(x+\frac{\theta}{2})\cdot\theta} = \frac{\sin(\frac{\theta}{2})}{\frac{\theta}{2}} \to 1$$

where,denominator is nothing but the area of cos(x), denoted by the integral $\int_a^b \cos(x)dx = \sin(b) - \sin(a)$(The left integral is nothing but the area of $\cos(x)$).
We now copy from $y = \sin(x)$ to $y = f(x)$: Let the height of $f(x)$ over the sub-interval $[x, x+\theta]$, $= f(x+\theta) - f(x)$, i.e. the Height of RT. Furthermore, the bottom of RT is θ, so the Slope of RT, $\frac{height}{bottom} = \frac{f(x+\theta)-f(x)}{\theta}$. Don't forget that the Height of Rect=Slope of RT=$\frac{f(x+\theta)-f(x)}{\theta}$. When $\theta \to 0$, the Height of Rect becomes the slope $f'(x)$:$\frac{f(x+\theta)-f(x)}{\theta} \to f'(x)$.Hence,the Height of Rect forms the slope curve with its curved equation to be $y = f'(x)$. Without loss of generality, we suppose $f'(x) > 0$ (see the calculus card next page),then,put right $f'(x)$ to left

$$\frac{f(x+\theta)-f(x)}{f'(x)\cdot\theta} \to 1$$

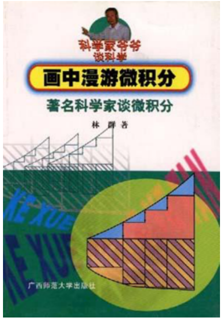
(or $\frac{Height\ of\ RT}{Area\ of\ Rect\ (with\ Height\ =\ Slope)} \to 1$).Summing up numerator and denominator

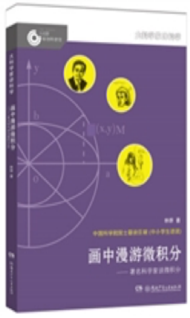
$$\frac{f(b)-f(a)}{\sum_x f'(x)\cdot\theta} \to 1$$

(or $\frac{total\ height}{total\ area} \to 1$, see the calculus card next page), i.e. the fundamental theorem

$\int_a^b f'(x)dx = f(b) - f(a)$(The left integral is nothing but the area of integrand)

I.e. determined by one-line(definition) and QED in two-lines(theorem). So, find out the derivative $f'(x)$, we obtain automatically the integral (or area) of the derivative. Thus the area is the byproduct of finding out derivative,"it's a waste if you don't learn it".
Rm: When Guangxi Normal University organized academicians to write Popular science books, Lin wrote the assumption of fundamental theorem and proof in 4 lines into it (See "Wander calculus through figures", 1998 and republished in 2017 by Hunan Children Press. Especially, the cover figure is nothing but the 2^{nd} figure)

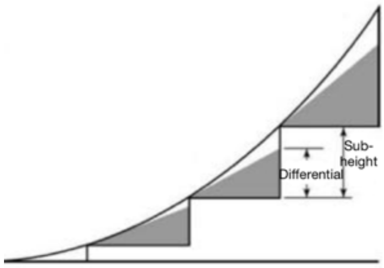




Arithmetic theorem is applied above:

$$\text{When denominators} > 0,\ \frac{\text{numerators}}{\text{denominators}} \to 1 \Rightarrow \frac{\text{sum of numerators}}{\text{sum of denominators}} \to 1$$

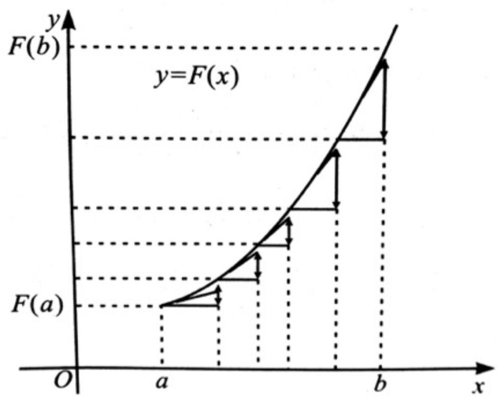
Here, what is called $\to 1$? denominators > 0 did it lose generality? See below:

Lin Calculus card (Three words are three knowledge points, numerator, denominator and 0.9)	
Arithmetic assumptions: The numerator of each fraction is sufficiently close to the denominator , that the division between them arbitrarily closes to 1, then: $0.999\ldots9 < \frac{\text{numerator}}{\text{denominator}} < \frac{1}{0.999\ldots9}$ Here, numerators and denominators are different.	Applied into calculus: let numerator = height of the small right triangle (called differential) denominator = height of the corresponding curved triangle (called sub-height) Let them satisfy arithmetic assumption (as long as it is assumption, proof is unnecessary) $0.999\ldots9 < \frac{\text{differential}}{\text{sub-height}} \left(= \frac{\text{slope of tangent}}{\text{slope of secant}}\right) < \frac{1}{0.999\ldots9}$
Assume denominators >0 and the number of 9 are the same (uniformity). Then, there is arithmetic theorem: $0.999\ldots9 < \frac{\text{sum of numerators}}{\text{sum of denominators}} < \frac{1}{0.999\ldots9}$ From part of the fraction to the integral fraction, uniformity is required.	(called uniformly differentiable). By arithmetic theorem 
It is only needed to test the left part: $0.999\ldots9 \times \text{denominator} < \text{numerator}$ $0.999\ldots9 \times \text{sum of denominators} < \text{sum of numerators}$ Similarly, test the right part. i.e. determined by one-line (arithmetic assumption) and QED in two-lines (arithmetic theorem)	$0.999\ldots9 < \frac{\text{sum of differential}}{\text{sum of sub-height}} < \frac{1}{0.999\ldots9}$ The number of 9 is increasing as the number of cutting points increase. i.e. sum of differentials $\to$ sum of sub-heights or integration of differentials = total height. This is called Calculus Fundamental Theorem

Explanation: Why is the number of 9 the same? If the number of occurrences of 9 is not enough, then encrypt its corresponding small base until the number of 9 is enough.
Besides, the assumption that denominators>0 don't lost generality. See exercise below.
Exercise: Assume $f'(x)$ is bounded (Necessary conditions) or $-M < f'(x)$. Let $F(x) = f(x) + (M+1)x$. Then, $F'(x) = f'(x) + M + 1 > 1$, (i.e. F is strongly monotone)which equals to: $\int_a^b f'(x)dx = f(b) - f(a)$ (easy to prove) E.g. $f(x) = \sin(x), F(x) = \sin(x) + 3x$

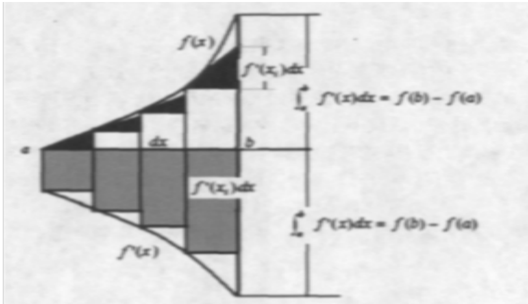
Key: Shortest proof-Definition of differential (or slope of tangent) has already contained fundamental theorem (by arithmetic theorem). i.e. As long as the definition or assumption is good, the theorem is acquired. Determined by one-line and QED in two-lines. It is so simple!

The figure in the above card can be explained by the following climbing.

<table>
<tr><td colspan="2">“mountain climbing image” of calculus. Look, don't think.</td></tr>
<tr><td>

Lin's Calculus image
P.235 in Jingzhong Zhang's book “Calculus without limit” and P.62 in “National scientific syllabus” (comments by You Chunguang and Xie Manting in National scientific syllabus)</td><td>When the mountain is very high, it is difficult to calculate its height. The mountain is winding. Thus, we can only measure the sub-height of pieces, and then add them up to obtain the total height. Even so, the height of each piece is still unknown. The mountain can be treated as a function whose specific form is still unknown. We make a tangent in the current position, replace the small section of the mountain by the tangent, and then calculate the height of the tangent line. Moreover, an error comes from replacing curved piece by the corresponding tangent. The smaller the piece is, the better approximation becomes. Furthermore, as the segment becomes smaller and smaller, the mountain's approximate height approaches the mountain's real height</td></tr>
</table>

Remark: Three Figs in Sect.1 merge in one Fig


$f(x)$
$f'(x_i)dx$
$\int_a^b f'(x)dx = f(b) - f(a)$
a
dx
b
$f'(x_i)dx$
$\int_a^b f'(x)dx = f(b) - f(a)$
$f'(x)$

Summary: fundamental theorem (2^{nd} line in the table below) turns to several arithmetic exercises (last 4 lines in the table below prove: loaf = pie)

	$\dfrac{\text{slope of secant}}{\text{slope of tangent}} \to 1$	$\dfrac{\text{sum of sub-heights}}{\text{sum of differentials}} \to 1$	determines on left hand side (definition); QED on right hand side (theorem). From high school exercise (last 4 lines) to generalized theorem (2nd line) is just "soft technique". It is applied above: division→1 (i.e. between $0.999\ldots$ and $\frac{1}{0.999...}$)
$f(x)$	$\dfrac{f(x+\theta)-f(x)}{f'(x)\cdot\theta}$	$\dfrac{f(b)-f(a)}{\sum\limits_x f'(x)\cdot\theta}$	
$\sin x$	$\dfrac{\sin(x+\theta)-\sin x}{\cos x\cdot\theta} = \dfrac{\text{sub-height}}{\text{differential}}$	$\dfrac{\sin(b)-\sin(a)}{\sum\limits_x \cos x\cdot\theta} = \dfrac{\text{total height (loaf)}}{\text{total area (pie)}}$	
$\tan x$	$\dfrac{\tan(x+\theta)-\tan(x)}{\frac{\theta}{\cos^2(x)}} = \dfrac{\text{sub-height}}{\text{differential}}$	$\dfrac{\tan(b)-\tan(a)}{\sum\limits_x \frac{\theta}{\cos^2(x)}} = \dfrac{\text{total height (loaf)}}{\text{total area (pie)}}$	
x^3	$\dfrac{(x+\theta)^3-x^3}{3x^2\theta} = \dfrac{\text{sub-height}}{\text{differential}}$	$\dfrac{b^3-a^3}{\sum\limits_x 3x^2\theta} = \dfrac{\text{total height (loaf)}}{\text{total area (pie)}}$	
$\sqrt{x}$	$\dfrac{\sqrt{x+\theta}-\sqrt{x}}{\frac{\theta}{2\sqrt{x}}} = \dfrac{\text{sub-height}}{\text{differential}}$	$\dfrac{\sqrt{b}-\sqrt{a}}{\sum\limits_x \frac{1}{2\sqrt{x}}\theta} = \dfrac{\text{total height (loaf)}}{\text{total area (pie)}}$	

Above is a cup of water of calculus, it utilized division→1, which is enough to beginner.

Below is meticulous processing. By using the traditional method: |subtraction|<<1 (i.e. between 0.999... – 1 and 1 – 0.999...). This is a barrel of water of calculus. Teachers should work hard on it. See appendix 1: The direct way of fundamental theorem.

These 2 ways achieve the same goal. They made “pie = loaf” more trustable (traditional way doesn’t require denominator>0).

Appendix 1 Classroom Life Experience: Direct Way of Fundamental Theorem

What is direct calculation? The formula $\sin b - \sin a = \int_a^b \cos x dx$ is calculated based on the concept of derivative. However, without that, we can also find this formula by trigonometric formula. We call such approach direct calculation.
Macro: mathematical proof (Omit)

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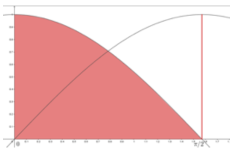
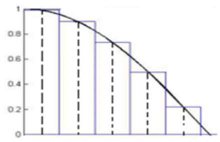
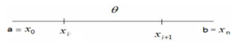
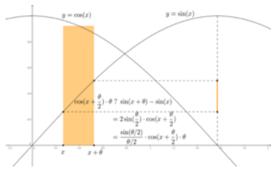
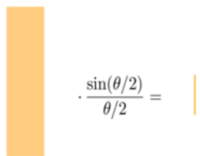
<td></td>
<td></td>
<td></td>
<td>Area of a rectangle = length × width
divide domain of the curve into small intervals evenly, length θ

θ decreases: $\theta \to 0$</td>
</tr>
<tr>
<td>Trigonometry: area below cos = height of sin</td>
<td>flooring for a curved classroom. Take straight lines as a substitute for a curve</td>
<td>Cover using rectangles</td>
<td>text description</td>
</tr>
<tr>
<td></td>
<td>
</td>
<td>Step 1, split:
The area of a sub-rectangle Corresponds to the length of a line segment $\frac{\sin\theta}{\theta} = 0.999\ldots$</td>
<td>sub-height of sin :
= $0.999\ldots\times$ area of small rectangle under cos</td>
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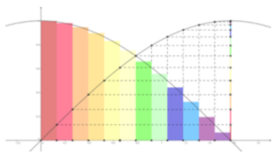
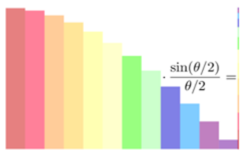
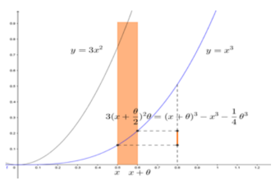
<td rowspan="2"></td>
<td rowspan="2">
</td>
<td>Step 2, merge:
The summed area of all the rectangles corresponds to the length of the whole line segments</td>
<td>sum up: sub-height turns to total height of sin = $0.999\ldots\times$ sum of areas of small rectangle under cos</td>
</tr>
<tr>
<td>Step 3, subdivide:
The whole space and the total area “goes to” the whole length of the line segment</td>
<td>area under cos (pie) = total height of sin (loaf) it is showed that: Start at sub-height, determined by one-line and QED in two-lines</td>
</tr>
<tr>
<td></td>
<td colspan="3">learn by analogy:
sub-height for x^3 = area of small rectangles under $3x^2$ + small tail
Since the sum of small tails is small, area under $3x^2$ (pie) = total height of x^3 (loaf)
general function is: sub-height of function A = areas of small rectangles under function B + small tail
Since the sum of small tails is small, area under function B (pie) = total height of function A (loaf).
All above starts from sub-height, determined by one-line and QED in two-lines.
(images credit to Yu Li)</td>
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Micro: mathematical proof

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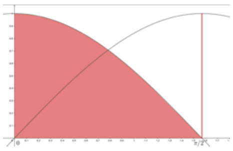
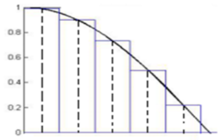
<td></td>
<td></td>
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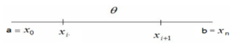
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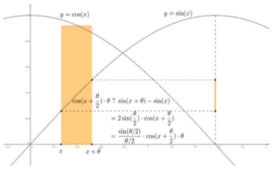
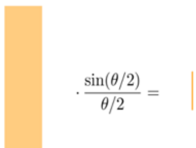
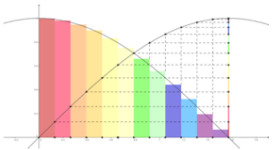
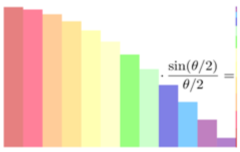
<td>trigonometry:
area below = height of sin</td>
<td>flooring for a curved classroom. Take straight lines as a substitute for a curve</td>
<td>Cover using rectangles</td>
<td>mathematical proof:
direct calculation</td>
</tr>
<tr>
<td></td>
<td>
</td>
<td>Step 1, split:
The area of a sub-rectangle corresponds to the length of a line segment.</td>
<td>sub-height from x to $x+\theta$ of sin $\leftrightarrow$ area of small rectangle of cos: $\sin(x+\theta)-\sin(x) = \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} \cdot \cos(x+\frac{\theta}{2}) \cdot \theta$
factor = 0.999...</td>
</tr>
<tr>
<td rowspan="2"></td>
<td rowspan="2">
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<td>Step 2, merge:
The summed area of all the rectangles corresponds to the length of the whole line segment.</td>
<td rowspan="2">Total height from x to $x+\theta$ of sin $\leftrightarrow$
sum of area of small rectangle of cos:
$\sin(b)-\sin(a) = \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} \cdot \sum_x \cos(x+\frac{\theta}{2}) \cdot \theta$
factor = 0.999...
Hence, when $\theta \to 0$, sum of right hand side turns to left hand side, which is defined as area under cos, denoted as: $\int_a^b \cos x dx = \sin b - \sin a$ it is showed that: area under cos (pie) = total height of sin (loaf) Start at sub-height, determined by one-line and QED in two lines</td>
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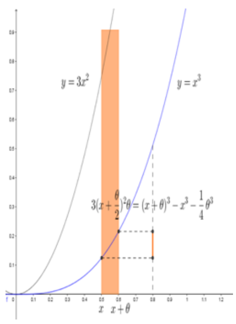


Yu Li computed $x^3, \sqrt{x}$;
Hongtao Chen computed $\tan(x)$;
Yan Gan computed $e^x, \ln x$.

Trigonometry turns to algebra x^3, small tail shows up (unrelated to x)
$(x+\theta)^3 - x^3 = 3(x+\frac{\theta}{2})^2\theta + \frac{1}{4}\theta^3$
$b^3 - a^3 = \sum_x 3(x+\frac{\theta}{2})^2\theta + \frac{1}{4}\theta^2(b-a)$
$\to$ area of $3x^2$ (when $\theta \to 0$)

$$\int_a^b 3x^2\,dx = b^3 - a^3$$

It is showed that area under $3x^2$ = height of x^3, and start at sub-height, determined by one-line and QED in two-lines

General expansion differences on that tail includes x:

$$(x+\theta)^3 - x^3 = 3x^2\theta + \text{tail}\cdot\theta$$
$$,\ \text{tail} = 3x\theta + \theta^2$$

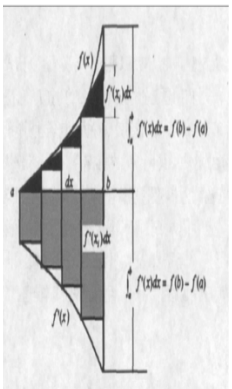
We have to use inequality to erase x:

$$|\text{tail}| \le \text{upper bound}(\theta) = 3b\theta + \theta^2 \quad (1)$$

$\Rightarrow b^3 - a^3 = \sum_x 3x^2\theta + \sum_x \text{tail}\cdot\theta$
,where sum of small tail (= tail$\cdot\theta$) is small:

$$|\sum_x \text{tail}\cdot\theta| \le \sum_x |\text{tail}|\cdot\theta$$
$$\le \text{upper bound}(\theta)\cdot\sum_x \theta$$
$$= (3b\theta+\theta^2)\cdot(b-a) \to 0 (2)$$
$$\Rightarrow b^3 - a^3 = \int_a^b 3x^2\,dx \qquad (3)$$

The above two algebra computations, determined by (1) (seeking sub-height) and QED in (3) (seeking full height), although each line needs be explained.



Generalization: generalize x^3 in exercise into general function $f(x)$. Denote the first term of expansion ($3x^2$) as $g(x)$ or $f'(x)$:

$$f(x+\theta) - f(x) = g(x)\cdot\theta + \text{tail}(x,\theta)\cdot\theta$$

the tail above is a function of x. There need to be an inequality to erase x:

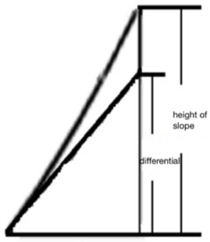
$$|\text{tail}(x,\theta)| \le \text{upper bound}(\theta) \to 0 \qquad (0.1)$$

(This is assumption ($\Leftrightarrow$*each points* $\to$*0, see book Lax etc.*) $\therefore$ proof is not needed)
$\Rightarrow$ conclusion: $f(b) - f(a) = \sum_x g(x)\theta + \sum_x \text{tails}(x,\theta)\cdot\theta$
, where sum of small tails (= tail $\cdot\theta$) is small:

$$|\sum_x \text{tails}(x,\theta)\cdot\theta| \le \sum_x |\text{tails}(x,\theta)|\cdot\theta \le \text{upper bound}(\theta)\cdot\sum_x\theta$$
$$= \text{upper bound}(\theta)\cdot(b-a) \to 0 \qquad (0.2)$$

<table>
<tr><td></td><td>$$\Rightarrow \text{fundamental theorem:} f(b) - f(a) = \int_a^b g(x)dx \qquad (0.3)$$

It is showed that area under g = height of f, and definition (0.1) determines theorem (0.3). i.e. “determined by one-line and QED in two-lines”

Above is also called teaching by exercise: show the exercise first. Acquire theorem from exercise. Since exercise is determined in 1 line, theorem is also determined in 1 line; Since exercise is solved in 2 lines, theorem is also solved in 2 lines. This may be the shortest proof.</td></tr>
<tr><td>
</td><td>Why $g(x) = f'(x)$? Observe (0.1), divide θ:

$$\frac{f(x+\theta) - f(x)}{\theta} = g(x) + \text{tail}(x,\theta) \to g(x)(\text{when } \theta \to 0$$
left hand side is: slope of secant from x to $x+\theta$ for curve $f(x)$
right hand side $g(x)$ is: limit of slope of secant, without θ, it is defined as slope of tangent at point x for curve $f(x)$, denoted as $f'(x)$, which is unique.</td></tr>
</table>

It is showed that a theorem is crystallization from exercises (shards) using the technique of imagining, trying, testing, and guessing. i.e. Apply exercises (shards) to conclude theorem. Textbooks do the opposite: apply the theorem to solve exercises. It concealed the truth of invention, making students focus on solving problems only without knowing how do theorems come. Students would think theorems as premises of exercises rather than that exercises as premises of theorems. If an educational revolution is needed, then, textbooks need to be reformed. What’s the point of learning mathematics? Students from primary school and middle school would say that “mathematics could be used in perimeter and area of a polygon.” But for curl graph (such as a circle), they wouldn’t know how to solve it. Or we should say students only know the formula without seeing why the formula is correct. This is where calculus is needed. The key point is fundamental theoremąłbest tool for advanced mathematics. Luckily, this tool can be grasped in several pages, including strict proof. So, this is the most economical and powerful way to learn calculus. In several pages, not only mathematical knowledge gets spread, but also real problems get solved, including Taylor formula, the most important exercise. This way costs little, gains much.

APPENDIX 2 FUNDAMENTAL THEOREM COSTS LITTLE, GAINS MUCH

From the fundamental theorem, other theorems become exercises (Thus, only one theorem) or by-products, even though the later are very useful.
Exercise 1 The sign of derivative ⇒ monotonicity of function (without using the mean value theorem)

$$f'(x) \begin{array}{l} >\theta \\ =\theta \\ <\theta \end{array} \Rightarrow \begin{array}{ll} f & \uparrow \\ f & =c \\ f & \downarrow \end{array}$$

Exercise 2 Fundamental theorem ⇒ Taylor formula
Use iterated integral, rather than multiple integral. Each step is no more than four lines:

First order:

$$\int_0^s f'(x)dx = f(s) - f(0)$$

Second order:

$$\int_0^{s_1}\int_0^{s_2} f''(x)dxds_2 = \int_0^{s_1}[f'(s_2) - f'(0)]ds_2 = f(s_1) - f(0) - f'(0)s_1$$

Thrid Order:

$$\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} f'''(x)dxds_3ds_2 = \int_0^{s_1}\int_0^{s_2}[f''(s_3) - f''(0)]ds_3ds_2$$

$$= \int_0^{s_1}[f'(s_2) - f'(0) - f''(0)s_2]ds_2 = f(s_1) - f(0) - f'(0)s_1 - \frac{1}{2}f''(0){s_1}^2$$

Order n+1:

$$\int_0^{s_1}\cdots\int_0^{s_n}\int_0^{s_{n+1}} f^{(n+1)}(x)dxds_{n+1}\cdots ds_2 = f(s_1) - f(0) - f'(0)s_1 - \cdots - \frac{1}{n!}f^{(n)}(0){s_1}^n$$

or

$$f(s_1) = f(0) + f'(0)s_1 + \cdots + \frac{1}{n!}f^{(n)}(0){s_1}^n + \int_0^{s_1}\cdots\int_0^{s_n}\int_0^{s_{n+1}} f^{(n+1)}(x)dxds_{n+1}\cdots ds_2$$

The last term cannot be calculated, and can be ignored in general (The absolute truth is unknown or too complicated, and thus should be replaced by the relative truth (i.e. the polynomials)). The Taylor formula simplifies elementary functions (complicated to calculate) as polynomials($+ - \times \div$). The simplest cases include:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, x \in \mathbb{R} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, x \in \mathbb{R}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots, x \in [-1, 1]$$

Especially, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Exercise 3 Fundamental theorem + Taylor formula $\Rightarrow$ circumference + circle area + ellipse area

Exercise 4 Differential equation: find $f(x)$ such that $f'(x) = g(x)$ or $f(0) = f_0$

$\Rightarrow \int_0^x g(t)dt = \int_0^x f'(t)dt = f(x) - f(0) \Rightarrow f(x) = f_0 + \int_0^x g(t)dt$

(See “Differential Equation and Triangulation” 2005 Tsinghua Press)

http://blog.sciencenet.cn/home.php?mod=attachment&id=65112

《A Great Way To Care II》